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A New Asymmetric Long-Range Model and Algebraic Bethe Ansatz

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A new integrable long-range model is derived from a new asymmetric R-matrix recently discussed by Bibikov in relation to a XXZ spin chain in an external magnetic field. The algebraic Bethe Ansatz is used to derive the eigenvalues and equations for the eigen momenta both for the usual and long-range model.

1. INTRODUCTION

In the theory of quantum integrable systems, quantum spin chains have always occupied a special status, since they were amongst the first systems to be solved exactly. Recently in a short note Bibikov (2000) has presented an R-matrix for the asymmetrical XXZ spin chain placed in an external magnetic field. The system was also analyzed in Albertini *et al.* (1997). In terms of the R-matrix the analysis of such a system can be accomplished using the algebraic Bethe Ansatz. The R-matrix obtained in Albertini *et al.* (1997) presents certain novel features in the sense that the spectral parameters on which the R-matrix depends are two-component vectors. The inclusion of the external magnetic field within the R-matrix itself is facilitated by assigning one component of this vector valued spectral parameter to be proportional to the external magnetic field. In this communication we will discuss the diagonalization problem for the asymmetric XXZ chain, which will be useful in the derivation of a long-range integrable Hamiltonian system from this spin system through the introduction of inhomogeneties. Such long-range models were first obtained in Hikami *et al.* (1992) for the XXZ spin system in the absence of external magnetic fields and have also been studied in Choudhury and Chowdhury (1996) and de Vega (1984).

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2. FORMULATION

We consider the R-matrix presented in Bibikov (2000) in a slightly different form as

$$\tilde{R}(\vec{\lambda}, \vec{\mu}) = \begin{pmatrix} a(\lambda_1, \mu_1)\, e^{\eta(\lambda_2-\mu_2)} & 0 & 0 & 0 \\ 0 & b(\lambda_1, \mu_1)\, e^{-\eta(\lambda_2+\mu_2)} & sh\eta & 0 \\ 0 & sh\eta & b(\lambda_1, \mu_1)\, e^{-\eta(\lambda_2+\mu_2)} & 0 \\ 0 & 0 & 0 & a(\lambda_1, \mu_1)\, e^{-\eta(\lambda_2-\mu_2)} \end{pmatrix} \tag{1.1}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2)$ and $\vec{\mu} = (\mu_1, \mu_2)$ are two-component vectors and

$$a(\lambda_1, \mu_1) = sh(\lambda_1 - \mu_1 + \eta), \qquad b(\lambda_1, \mu_1) = sh(\lambda_1 - \mu_1)$$

η being a quantization parameter.

This R-matrix satisfies the Yang–Baxter equation (Korepin *et al.*, 1993):

$$\tilde{R}_{12}(\vec{\lambda}, \vec{\mu})\tilde{R}_{13}(\vec{\lambda}, \vec{\nu})\tilde{R}_{23}(\vec{\mu}, \vec{\nu}) = \tilde{R}_{23}(\vec{\mu}, \vec{\nu})\tilde{R}_{13}(\vec{\lambda}, \vec{\nu})\tilde{R}_{12}(\vec{\lambda}, \vec{\mu}) \tag{1.2}$$

It is interesting to note that Eq. (1.2) remains valid even after we introduce inhomogeneties $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ by means of the transformations $\vec{\lambda} \to \vec{\lambda} - \vec{\alpha}$, $\vec{\mu} \to \vec{\mu} - \vec{\beta}$, $\vec{\nu} \to \vec{\nu} - \vec{\gamma}$. The connection with the local Hamiltonian density of the asymmetric XXZ spin chain in presence of an external magnetic field can be made by employing the following reduction.

Let $\vec{\lambda} = (\lambda, \lambda h)$, $\vec{\nu} = (\nu, \nu h)$, where h is the magnetic field. Defining

$$\tilde{R}(\lambda, \nu) \equiv L(\lambda, \nu)$$

we have

$$L(\lambda, \nu) = \begin{pmatrix} sh(\lambda-\nu+\eta)\, e^{\eta h(\lambda-\nu)} & 0 & 0 & 0 \\ 0 & sh(\lambda-\nu)\, e^{-\eta h(\lambda+\nu)} & sh\eta & 0 \\ 0 & sh\eta & sh(\lambda-\nu)\, e^{\eta h(\lambda+\nu)} & 0 \\ 0 & 0 & 0 & sh(\lambda-\nu+\eta)\, e^{-\eta h(\lambda-\nu)} \end{pmatrix} \tag{1.3}$$

Now it can be shown that the local Hamiltonian density follows from

$$H_{i,i+1} = sh\eta L^{-1}(\lambda, \nu)\frac{\partial}{\partial\lambda}L(\lambda, \nu)\,|_{\lambda=\nu} - \frac{ch\eta}{2}I_4 \tag{1.4}$$

$$= \frac{ch\eta}{2}\sigma_i^3 \otimes \sigma_{i+1}^3 + \frac{h\eta sh\eta}{2}\left(\sigma_i^3 \otimes I + I \otimes \sigma_{i+1}^3\right) + e^{-h\psi}\sigma_i^- \otimes \sigma_{i+1}^+ + e^{h\psi}\sigma_i^+ \otimes \sigma_{i+1}^- \tag{1.5}$$

where $\psi = 2\eta\nu$ and we have written $L(\lambda, \nu)$ in terms of local operators $(\vec{\sigma}_i, \vec{\sigma}_{i+1})$ which are the 2×2 Pauli matrices. As shown in Bibikov (2000) the L operator satisfies the standard relation of Quantum Inverse Scattering Method:

$$\hat{R}(\lambda, \mu)L(\lambda, \nu) \otimes L(\mu, \nu) = L(\mu, \nu) \otimes L(\lambda, \nu)\hat{R}(\lambda, \mu) \tag{1.6}$$

where

$$\hat{R}(\lambda, \mu) = P\tilde{R}(\lambda, \mu) \tag{1.7}$$

P being the usual permutation operator satisfying $P(\vec{x} \otimes \vec{y}) = \vec{y} \otimes \vec{x}$, where $\vec{x}, \vec{y} \in \mathbb{C}$. Equation (1.6) will be of crucial importance in our subsequent analysis. First it will be noticed that ν in Eq. (1.6) plays the role of an auxillary parameter. Let us now consider a one-dimensional lattice and assign to each site the operator

$$\hat{L}(\lambda, \nu_k) \equiv R_{0k}(\lambda, \nu_k; \eta) = \frac{1}{sh(\lambda - \nu_k + \eta)}\tilde{R}_{0k}(\lambda, \nu_k; \eta) \tag{1.8}$$

where the indices 0 and k refer to an auxiliary space V_A and a quantum space V_k at the kth lattice site respectively. Thus $\hat{L}(\lambda, \nu_k)$ as defined above is now defined in the tensor product of two vector space i.e. $V_A \otimes V_k$. Notice that the parameter ν has now become site-dependent. We mention two properties of $R_{0k}(\lambda, \nu_k; \eta)$ which will be frequently used in the following analysis.

$$\text{(i) } R_{0k}(\nu_k, \nu_k; \eta) = P_{0k}; \qquad \text{(ii) } R_{0k}(\lambda, \nu_k; \eta = 0) = I \tag{1.9}$$

Next we define a transition matrix

$$T_N(\lambda, \{\nu_k\}; \eta) = R_{0N}(\lambda, \nu_N; \eta) \cdots R_{0k}(\lambda, \nu_k; \eta) \cdots R_{01}(\lambda, \nu_1; \eta) \tag{1.10}$$

In view of the local relation (1.6) the transition matrix defined above is also intertwined by $\hat{R}(\lambda, \mu)$, i.e.

$$\hat{R}(\lambda, \mu)T_N(\lambda, \{\nu_k\}; \eta) \otimes T_N(\mu, \{\nu_k\}; \eta) = T_N(\mu, \{\nu_k\}; \eta) \otimes T_N(\lambda, \{\nu_k\}; \eta)\hat{R}(\lambda, \mu) \tag{1.11}$$

As is well known the trace of the transition operator is a generator of the integrals of motion and is called the transfer matrix $t(\lambda)$, i.e.

$$t(\lambda, \{\nu_k\}; \eta) = \mathrm{tr}_0\{R_{0N}(\lambda, \nu_N; \eta) \cdots R_{0k}(\lambda, \nu_k; \eta) \cdots R_{01}(\lambda, \nu_1; \eta)\} \tag{1.12}$$

From Eq. (1.11) it immediately follows that

$$[t(\lambda), t(\mu)] = 0 \quad \text{for all } \lambda, \mu \tag{1.13}$$

We are now in a position to derive the eigenvalue of the transition matrix, using algebraic Bethe Ansatz.

3. ALGEBRAIC BETHE ANSATZ

Let us write the transition matrix in the following form:

$$T(\lambda, \{\nu_k\}; \eta) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix} \tag{2.1}$$

Then from Eq. (1.11) we pick up the following two commutation relations:

$$\begin{aligned} A_N(\lambda)C_N(\mu) &= \frac{sh(\lambda-\mu+\eta)\, e^{\eta h(\lambda-\mu)}}{sh(\lambda-\mu)\, e^{-\eta h(\lambda+\mu)}} C_N(\mu)A_N(\lambda) \\ &\quad - \frac{sh(\eta)}{sh(\lambda-\mu)\, e^{-\eta h(\lambda+\mu)}} C_N(\lambda)A_N(\mu) \end{aligned} \tag{2.2}$$

$$\begin{aligned} D_N(\lambda)C_N(\mu) &= \frac{sh(\mu-\lambda+\eta)\, e^{\eta h(\mu-\lambda)}}{sh(\mu-\lambda)\, e^{-\eta h(\mu+\lambda)}} C_N(\mu)D_N(\lambda) \\ &\quad - \frac{sh(\eta)}{sh(\mu-\lambda)\, e^{-\eta h(\mu+\lambda)}} C_N(\lambda)D_N(\mu) \end{aligned} \tag{2.3}$$

We should mention here that $C_N(\lambda)$ is considered as a creation operator so that an M-excitation eigenstate of the transfer matrix given by Eq. (1.12) would be of the form

$$\Omega_M = C_N(\mu_1)C_N(\mu_2)C_N(\mu_3)\cdots C_N(\mu_N)|0\rangle \tag{2.4}$$

where the vacuum state is being defined in the following manner:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N \tag{2.5}$$

Application of $A_N(\lambda)$ to the state Ω_M results in the following equation upon using (2.2) repeatedly:

$$A_N(\lambda)\Omega_M = a_N(\lambda) \prod_{i=1}^{M} \frac{sh(\lambda-\mu_i+\eta)}{sh(\lambda-\mu_i)} e^{2h\eta\lambda}\Omega_M + \text{unwanted terms} \tag{2.6}$$

where by a "unwanted terms" we designate all those terms which are not proportional to Ω_M. Similar application of $D_N(\lambda)$ to Ω_M using Eq. (2.3) results in the following:

$$D_N(\lambda)\Omega_M = d_N(\lambda) \prod_{i=1}^{M} \frac{sh(\mu_i-\lambda+\eta)}{sh(\mu_i-\lambda)} e^{2h\eta\lambda} + \text{unwanted terms} \tag{2.7}$$

In Eqs. (2.6) and (2.7) $a_N(\lambda)$ and $d_N(\lambda)$ are the eigenvalues of $A_N(\lambda)$ and $D_N(\lambda)$ when acting on $|0\rangle$. These eigenvalues may be determined by noting that in general

$$(T_N(\lambda))_{ab} = \sum_{a_1,a_2,\ldots,a_{N-1}=1}^{2} t^{(1)}_{aa_1}(\lambda, \nu_1) t^{(2)}_{a_1a_2}(\lambda, \nu_2) \cdots t^{(N)}_{a_{N-1}b}(\lambda, \nu_N) \tag{2.8}$$

where $t^{(k)}_{ij}(\lambda, \nu_k)$ refer to the partitioned submatrix of $R_{0k}(\lambda, \nu_k; \eta)$; that is we write

$$R_{0k}(\lambda, \nu_k; \eta) = \begin{pmatrix} t^k_{11}(\lambda, \nu_k) & t^k_{12}(\lambda, \nu_k) \\ t^k_{21}(\lambda, \nu_k) & t^k_{22}(\lambda, \nu_k) \end{pmatrix} \tag{2.9}$$

with

$$\begin{aligned}
t^k_{11}(\lambda, \nu_k) &= \begin{pmatrix} e^{\eta h(\lambda-\nu_k)} & 0 \\ 0 & \frac{sh(\lambda-\nu_k)}{sh(\lambda-\nu_k+\eta)}\, e^{-\eta h(\lambda+\nu_k)} \end{pmatrix} \\
t^k_{12}(\lambda, \nu_k) &= \begin{pmatrix} 0 & 0 \\ \frac{sh(\eta)}{sh(\lambda-\nu_k+\eta)} & 0 \end{pmatrix} \\
t^k_{21}(\lambda, \nu_k) &= \begin{pmatrix} 0 & \frac{sh(\eta)}{sh(\lambda-\nu_k+\eta)} \\ 0 & 0 \end{pmatrix} \\
t^k_{22}(\lambda, \nu_k) &= \begin{pmatrix} \frac{sh(\lambda-\nu_k)}{sh(\lambda-\nu_k+\eta)}\, e^{\eta h(\lambda+\nu_k)} & 0 \\ 0 & e^{-\eta h(\lambda-\nu_k)} \end{pmatrix}
\end{aligned} \tag{2.10}$$

Applying the definition (2.8) we find that

$$A_N(\lambda)|0\rangle = \left(\prod_{k=1}^{N} e^{\eta h(\lambda-\nu_k)} \right) |0\rangle = a_N(\lambda)|0\rangle \tag{2.11}$$

$$D_N(\lambda)|0\rangle = \left\{ \prod_{k=1}^{N} \frac{sh(\lambda-\nu_k)}{sh(\lambda-\nu_k+\eta)}\, e^{\eta h(\lambda+\nu_k)} \right\} |0\rangle = d_N(\lambda)|0\rangle \tag{2.12}$$

so that $a_N(\lambda)$ and $d_N(\lambda)$ can be read off easily.

From Eqs. (2.6) and (2.7) it is obvious that Ω_M will be an eigenstate of $t(\lambda) \equiv \mathrm{tr}(T_N(\lambda))$ if the unwanted terms vanish. The vanishing of the unwanted terms leads to the Bethe Ansatz equations (BAEs) which determine the $\{\mu_i\}$'s. Under this condition one has

$$\begin{aligned}
t(\lambda)\Omega_M &= (A_N(\lambda) + D_N(\lambda))\Omega_M \\
&= \left[a_N(\lambda) \prod_{i=1}^{M} \frac{sh(\lambda-\mu_i+\eta)}{sh(\lambda-\mu_i)} e^{2\lambda\eta h} \right. \\
&\qquad \left. + d_N(\lambda) \prod_{i=1}^{M} \frac{sh(\mu_i-\lambda+\eta)}{sh(\mu_i-\lambda)} e^{2\lambda\eta h} \right] \Omega_M
\end{aligned} \tag{2.13}$$

while the vanishing condition for the unwanted terms is the following set of coupled equation:

$$\frac{a_N(\mu_j)}{d_N(\mu_j)} = \prod_{i\neq j}^{M} \frac{sh(\mu_i - \mu_j + \eta)}{sh(\mu_i - \mu_j - \eta)}, \quad j = 1, 2, \ldots, M \tag{2.14}$$

From (2.11) and (2.12) we find that the ratio on the left is

$$\frac{a_N(\mu_j)}{d_N(\mu_j)} = \prod_{k=1}^{N} \frac{sh(\mu_j - \nu_k + \eta)}{sh(\mu_j - \eta_k)} e^{-2\eta\nu_k h} \tag{2.15}$$

Hence the BAE finally assumes the following form:

$$\prod_{k=1}^{N} \frac{sh(\mu_j - \nu_k + \eta)}{sh(\mu_j - \eta_k)} e^{-2\eta\nu_k h} = \prod_{i=1, i\neq j}^{M} \frac{sh(\mu_i - \mu_j + \eta)}{sh(\mu_i - \mu_j - \eta)}, \quad j = 1, 2, \ldots, M \tag{2.16}$$

Furthermore the energy eigenvalue of the transfer matrix is seen to be given by

$$\Lambda_M(\lambda, \nu_k) = e^{2\eta\lambda h} \left(\prod_{k=1}^{N} e^{\eta h(\lambda - \nu_k)} \right) \prod_{i=1}^{M} \frac{sh(\lambda - \mu_i + \eta)}{sh(\lambda - \mu_i)} + e^{2\eta\lambda h} \left(\prod_{i=1}^{N} \frac{sh(\lambda - \nu_k)}{sh(\lambda - \nu_k + \eta)} e^{\eta h(\lambda + \nu_k)} \right) \prod_{i=1}^{M} \frac{sh(\mu_i - \lambda + \eta)}{sh(\mu_i - \lambda)} \tag{2.17}$$

Equation (2.16) and (2.17) complete the Bethe Ansatz solution of the eigenvalue problem:

$$t(\lambda)\Omega_M = \Lambda_M \Omega_M$$

Analysis of the BAE itself is a nontrivial exercise which we shall not go into here.

4. LONG-RANGE HAMILTONIAN

To deduce a family of Hamiltonians exhibiting long-range interaction we shall consider the following eigenvalue problem:

$$Z_k|\Omega\rangle = \prod_{l=1}^{M} \varepsilon(\mu_l \sim \nu_k)|\Omega\rangle = \zeta_k|\Omega\rangle \tag{3.1}$$

where the state $|\Omega\rangle$ is characterized by a set of M quasi-momenta $\{\mu_l\}_{l=1}^{M}$ and Z_k is defined as follows:

$$Z_k = t(\lambda = \nu_k, \{\nu\}_{p=1}^{N}; \eta) \tag{3.2}$$

Recalling the definition of $t(\lambda)$ as given by Eq. (1.12) we see that

$$Z_k = \mathrm{tr}_0\{R_{0N}(\nu_k, \nu_N; \eta) \cdots R_{0k+1}(\nu_k, \nu_{k+1}; \eta) \times P_{0k} R_{0k-1}(\nu_k, \nu_{k-1}; \eta) \cdots R_{01}(\nu_k, \nu_1; \eta)\} \tag{3.3}$$

with

$$[Z_k, Z_l] = 0 \quad \text{for all } l, m \tag{3.4}$$

by making use of Eq. (1.9). Furthermore it can be easily verified using (1.9) again that

$$Z_k|_{\eta=0} = I_N \otimes I_{N-1} \cdots \otimes I_1 = I \tag{3.5}$$

Differentiating Eq. (3.1) w.r.t η and setting $\eta = 0$ gives us

$$\left(\frac{\partial Z_k}{\partial \eta}\right) |\Omega\rangle\,|_{\eta=0} + Z_k \frac{\partial}{\partial \eta} |\Omega\rangle\,|_{\eta=0} = \left(\frac{\partial}{\partial \eta} \prod_{l=1}^{M} \varepsilon(\mu_l \sim \nu_k)\right) |\Omega\rangle\,|_{\eta=0} + \prod_{l=1}^{M} \varepsilon(\mu_l \sim \nu_k) \frac{\partial}{\partial \eta} |\Omega\rangle\,|_{\eta=0} \tag{3.6}$$

Hence if we demand

$$\prod_{l=1}^{M} \varepsilon(\mu_l \sim \nu_k)|_{\eta=0} = 1 \tag{3.7}$$

then by making use of Eq. (3.5) one finds that

$$\left(\frac{\partial Z_k}{\partial \eta}\right) |\Omega\rangle|_{\eta=0} = \left(\frac{\partial}{\partial \eta} \prod_{l=1}^{M} \varepsilon(\mu_l \sim \nu_k)\right) |\Omega\rangle|_{\eta=0} \equiv \xi_k |\Omega\rangle \tag{3.8}$$

Now for small values of the quantization parameter η one finds that

$$R(\lambda, \mu) \to_{\eta \to 0} P[I + \eta r(\lambda, \mu) + 0(\eta)] \tag{3.9}$$

Consequently Z_k as defined by Eq. (3.3) can be shown to admit a power series expansion in η so that

$$Z_k = I + \eta H_k + 0(\eta) \tag{3.10}$$

where

$$H_k = \frac{\partial Z_k}{\partial \eta}|_{\eta=0} \tag{3.11}$$

In view of Eq. (3.4) one concludes that $[H_k, H_l] = 0$ so that Eq. (3.8) defines a eigenvalue problem for the commuting family of Hamiltonians $H_k (k = 1, 2, \ldots, N)$.

The explicit evaluation of Z_k from Eq. (3.4) can be performed following Hikami *et al.* (1992) so that we shall only give the final result, namely

$$Z_k = R_{kk-1}(\nu_k, \nu_{k-1}; \eta) \cdots R_{k1}(\nu_k, \nu_1; \eta) \\ \times R_{kN}(\nu_k, \nu_N; \eta) \cdots R_{kk+1}(\nu_k, \nu_{k+1}; \eta) \quad (3.12)$$

so that

$$H_k = \sum_{j=1 \neq k}^{N} \frac{\partial R_{kj}}{\partial \eta}(\nu_k, \nu_j; \eta)|_{\eta=0} \quad (3.13)$$

Using Eqs. (2.9) and (2.10) we can explicitly evaluate H_k given above and find that it may be expressed in terms of local spin operators in the form

$$H_k = \sum_{j \neq k}^{N} \frac{1}{2sh(\nu_k - \nu_j)} \big[\cosh(\nu_k - \nu_j)\big(\sigma_k^3 \otimes \sigma_j^3 - 1\big) + \big(\sigma_k^+ \otimes \sigma_j^- + \sigma_k^- \otimes \sigma_j^+\big) \\ + h(\nu_k - \nu_j) sh(\nu_k - \nu_j)\big(\sigma_k^3 \otimes \sigma_j^3 + 1 \otimes 1\big) \\ + h(\nu_k + \nu_j) sh(\nu_k - \nu_j)\big(1 \otimes \sigma_j^3 - \sigma_k^3 \otimes 1\big)\big] \quad (3.14)$$

Notice should be taken of the extra terms involving the external magnetic field h in (3.14); when $h = 0$ we get back the basic result of Hikami *et al.* (1992) for XXZ spin chain.

To ensure that the eigenvalue problem given by Eq. (3.8) is well defined we have to ensure that condition (3.7) is fulfilled. It is clear that the eigenvalue of Z_k when acting on the M-excitation state $|\Omega\rangle$ will be the same at that of $t(\lambda)$ evaluated at $\lambda = \nu_k$. From Eq. (2.17) we find that

$$\Lambda_M(\lambda = \nu_k) = e^{2\eta\nu_k h} \left(\prod_{j=1}^{N} e^{\eta h(\nu_k - \nu_j)} \right) \prod_{i=1}^{M} \frac{sh(\nu_k - \mu_i + \eta)}{sh(\nu_k - \mu_i)} \\ + e^{2\eta\nu_k h} \left(\prod_{j=1}^{N} \frac{sh(\nu_k - \nu_j)}{sh(\nu_k - \nu_j + \eta)} e^{\eta h(\nu_k + \nu_j)} \right) \\ \times \prod_{i=1}^{M} \frac{sh(\mu_i - \nu_k + \eta)}{sh(\mu_i - \nu_k)} \quad (3.15)$$

However the second term on the r.h.s. vanishes when $j = k$, and so

$$\Lambda_M(\lambda = \nu_k) = e^{2\eta\nu_k h} \left(\prod_{j=1}^{N} e^{\eta h(\nu_k - \nu_j)} \right) \prod_{i=1}^{M} \frac{sh(\nu_k - \mu_i + \eta)}{sh(\nu_k - \mu_i)} \quad (3.16)$$

Notice once again appearance of terms involving the external magnetic field in Eq. (3.16). Thus from Eqs. (3.16) and (3.1) we find that

$$\prod_{l=1}^{M} \varepsilon(\mu_l \sim \nu_k) = e^{2\eta\nu_k h} \left(\prod_{j=1}^{N} e^{\eta h(\nu_k - \nu_j)} \right) \prod_{l=1}^{M} \frac{sh(\nu_k - \mu_l + \eta)}{sh(\nu_k - \mu_l)} \tag{3.17}$$

In view of the constraint expressed by Eq. (3.7) one can see immediately that the r.h.s. of (3.17) reduces to unity when $\eta = 0$. Thus we conclude that the eigenvalue problem given by Eq. (3.8) is well defined. The eigenvalue corresponding to the Hamiltonian H_k can be easily deduced from Eq. (3.17) and is given by

$$\xi_k = h \left\{ 2\nu_k + \sum_{j=1}^{N} (\nu_k - \nu_j) \right\} + \sum_{l=1}^{M} \coth(\nu_k - \mu_l) \tag{3.18}$$

Finally the new BAE determining the μ_l's is obtained by differentiating Eq. (2.16) w.r.t. η and setting $\eta = 0$. This yields

$$\sum_{k=1}^{N} [\coth(\mu_l - \nu_k) - 2h\nu_k] = 2 \sum_{i \neq j}^{M} \coth(\mu_i - \mu_l) \quad l = 1, 2, \ldots, M \tag{3.19}$$

From Eq. (3.14) we see that the inhomogenities $\{\nu_k\}_{k=1}^{N}$ appear as coordinates in the expression for the Hamiltonian H_k. Moreover since the summation in Eq. (3.14) is not confined to nearest neighbors the Hamiltonian H_k exhibits long-range interactions in terms of the "coordinates" $\{\nu_k\}_{k=1}^{N}$. The external magnetic field appears in the form of additional terms in the expression for H_k. The energy eigenvalue equation (3.18) and the new BAE (3.19) both involve the external magnetic field. In the absence of the magnetic field the above results reduce to those derived earlier.

5. DISCUSSION

In our above analysis we have considered an asymmetric XXZ spin chain placed in an external magnetic field. The effect of the magnetic field can be incorporated within the standard framework of Quantum Inverse Scattering method, provided the spectral parameter is taken to be a two-component vector instead of the usual form of a complex scalar. The additional component of the spectral parameter takes into account the external magnetic field. We have deduced the eigenvalue of the transfer matrix for such a spin system by employing the standard technique of algebraic Bethe Ansatz, and have also deduced the corresponding Bethe Ansatz equation for the eigenvalue momenta. Finally we have considered an inhomogeneous version of this one-dimensional spin system in presence of a magnetic field and have derived a family of mutually commuting long-range Hamiltonian form it. This Hamiltonian explicitly contains terms involving the external magnetic field. The inhomogenity parameters take the role of pseudocoordinates of the long-range Hamiltonians. We have also derived the eigenvalues of these new Hamiltonians.

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